Remarks on Some Recent Papers Concerning the Computation of Coupled Coincidence Points

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Abstract—*In this note we first point out an omission in the hypotheses of Theorem 3.1, and then complete the proof of Theorem 4.1 of [1]. Similarly, we point out some omissions in the hypotheses of Theorem 4, and then complete the proof of Theorem 6 of [2]. Finally, we note that the results in [2] are not new but are the immediate consequences of the results proved in [1] and [3].*

Keywords*: Coupled coincidence points; partially ordered metric spaces; mixed monotone property; compatible mappings.*

2010 AMS Classification: Primary: 47H10; Secondary 54H25

1. INTRODUCTION

Theorem 3.1 of [1] (and Theorem 4 of [2]) establishes the existence of a coupled coincidence point within the framework of the partially ordered metric spaces. By adding additional hypotheses to Theorem 3.1 [1] (and Theorem 4 [2]), Theorem 4.1 of [1] (and Theorem 6 of [2], respectively) purports to show that the coupled coincidence point is unique. The reader should consult [1, 2, 3] for terms not specifically defined in this note.

Remark 1. The proof of Theorem 3.1 ([1], page 7, line 12) uses the fact that g is monotone increasing. The hypotheses of this theorem must also include this fact. The correct statement of Theorem 3.1 should now read as follows:

Theorem 1. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ and $g: X \to X$ be two mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preccurlyeq F(x_0, y_0) \text{ and } gy_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi(d(F(x, y),F(u, v))) \le \frac{1}{2}\,\phi(d(gx, gu)+d(gy, gv)) - \psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right),$$

for all $x, y, u, v \in X$ with $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$. Suppose $F(X \times X) \subseteq g(X)$; the mapping g is continuous, monotone increasing and compatible with F and also suppose either

(a) F is continuous, or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$, for all n;

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n.

Then there exist $x, y \in X$ such that

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x),$$

that is, F and g have a coupled coincidence point in X.

Remark 2. The proof of Theorem 4 ([2]) uses the fact that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$ and that the mapping F must have the mixed g-monotone property. The hypotheses of this theorem must also include these facts. Further, we note that the completeness of the range subspace g(X) relaxes the completeness of the space (X, d). The statement of Theorem 4 ([2]) should now read as follows:

Theorem 2. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X. Let $F: X \times X \to X$ and $g: X \to X$ be two mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preccurlyeq F(x_0, y_0) \text{ and } gy_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that F and g satisfy

$$\varphi(d(F(x, y),F(u, v))) \le \frac{1}{2}\,\varphi(d(gx, gu)+d(gy, gv)) - \psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right),$$

for all x, y, u, v ∈ X with $gx \preccurlyeq gu$ and $gy \succcurlyeq gv$. Suppose F(X × X) ⊆ g(X) g(X) is complete and the mapping g is continuous.

Suppose that either

(1) F is continuous, or
(2) X has the following property:
 (a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$, for all n ∈ ℕ;
 (b) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n ∈ ℕ.

Then there exist x, y ∈ X such that

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x),$$

that is, F and g have a coupled coincidence point in X.

Remark 3. We note that Theorem 2 (Theorem 4 [2]) is actually Theorem 2.11 of [3].

Remark 4. The proof of the Theorem 5 ([2], page 7, line 8) uses the fact that $gx \succcurlyeq gx_{n+1}$ and $gy \preccurlyeq gy_{n+1}$. This is possible since $\{gx_n\} \to gx$ and using condition (2) of the hypotheses of Theorem 5 [2]. Now, if we look at the proof of the Theorem 5 ([2], page 8, line 7), the authors use the fact that $\{gx_n\} \to x$. But this is not correct.

Remark 5. The conclusion of Theorem 4.1 (in [1]) is that F and g have a unique coupled coincidence point. However, the proof only shows that gx = gz and gy = gt, where (x, y) and (z, t) are assumed to be coupled coincidence points of F. It is necessary to show that x = z and y = t.

To complete the proof we shall first prove the following Lemma.

Lemma 1. Let F : X × X → X and g : X → X be compatible maps and there exists an element (x, y) ∈ X × X, such that gx = F(x, y) and gy = F(y, x), then gF(x, y) = F(gx, gy) and gF(y, x) = F(gy, gx).

Proof. Since the pair (F, g) is compatible, it follows that

$\lim_{n\to\infty} d(gF(x_n, y_n), F(g(x_n), g(y_n))) = 0,$
$\lim_{n\to\infty} d(gF(y_n, x_n), F(g(y_n), g(x_n))) = 0,$
whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = a$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = b$ for some a, b in X.
Taking $x_n = x$, $y_n = y$ and using the fact that gx = F(x, y), gy = F(y, x), it follows that
d(gF(x, y), F(gx, gy)) = 0 and d(gF(y, x), F(gy, gx)) = 0.
Hence, gF(x, y) = F(gx, gy) and gF(y, x) = F(gy, gx).

Now, we are ready to complete the proof of Theorem 4.1 [1].

Since (x, y) is a coupled coincidence point of F and g; that is, gx = F(x, y), gy = F(y, x); and the pair (F, g) is compatible, by Lemma 1, it follows that

$$ggx = gF(x, y) = F(gx, gy), \text{ and}$$
$$ggy = gF(y, x) = F(gy, gx). \qquad (1)$$

Denote gx = r, gy = s then by (1), we have

$$gr = F(r, s) \text{ and } gs = F(s, r). \qquad (2)$$

Thus, (r, s) is a coupled coincidence point of the mappings F and g. Then by (38) (in [1]) with z = r, t = s, it follows that

$$gr = r, \; gs = s. \qquad (3)$$

By (2) and (3), r = gr = F(r, s) and s = gs = F(s, r). Therefore, (r, s) is the coupled common fixed point of F and g. This proves the existence of coupled common fixed point of F and g. Also, if (p, q) is another coupled common fixed point of maps F and g, then by (38) (in [1]), we have p = gp = gr = r and q = gq = gs = s.

Remark 6. The above addition to the proof of Theorem 4.1 [1] not only proves the uniqueness of coupled coincidence point of F and g but also ensures the existence and uniqueness of coupled common fixed point of F and g.

Remark 7. The conclusion of Theorem 6 (in [2]) is that F and g have a unique coupled fixed point. However, the proof only shows that gx = gz and gy = gt, where (x, y) and (z, t) are assumed to be coupled coincidence points of F. It is necessary to show that x = z and y = t. The proof presented above along with Lemma 1, is also applicable to Theorem 6 [2], if in the hypotheses of Theorem 6 [2], we additionally assume that the mappings F and g are compatible.

Remark 8. We finally conclude that Theorems 4, 6 of [2] follows immediately from Theorem 2.11 of [3] and Theorem 4.1 of [1], respectively.

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